Algebraic Reasoning

Year 11

#MathsEveryoneCan

White Rose Maths

	Week 1	Week 2	Week 3	Week 4	Week 5	Week 6	Week 7	Week 8	Week 9	Week 10	Week 11	Week 12
Autumn	**Graphs**						**Algebra**					
	Gradients & lines		Non-linear graphs		Using graphs		Expanding & Factorising		Changing the subject		Functions	
Spring	**Reasoning**						**Revision and Communication**					
	Multiplicative		Geometric		Algebraic		Transforming & Constructing		Listing & describing		Show that...	
Summer	**Revision**						**Examinations**					

Spring 1 : Reasoning

Weeks 1 and 2: Multiplicative Reasoning

Students develop their multiplicative reasoning in a variety of contexts, from simple scale factors through to complex equations involving direct and inverse proportion. They link inverse proportion with the formulae for pressure and density. There is also the opportunity to review ratio problems.
National Curriculum content covered includes:

- compare lengths, areas and volumes using ratio notation and/or scale factors; make links to similarity
- understand that X is inversely proportional to Y is equivalent to X is proportional to $\frac{1}{Y}$
- **{construct and}** interpret equations that describe direct and inverse proportion
- extend and formalise their knowledge of ratio and proportion, including trigonometric ratios, in working with measures and geometry, and in working with proportional relations algebraically and graphically

Weeks 3 and 4: Geometric Reasoning

Students consolidate their knowledge of angles facts and develop increasingly complex chains of reasoning to solve geometric problems. Higher tier students revise the first four circle theorems studied in Year 10 and learn the remaining theorems. Students also revisit vectors and the key topics of Pythagoras' theorem and trigonometry.
National Curriculum content covered includes:

- reason deductively in geometry, number and algebra, including using geometrical constructions
- **{apply and prove the standard circle theorems concerning angles, radii, tangents and chords, and use them to prove related results}**
- interpret and use bearings
- apply addition and subtraction of vectors, multiplication of vectors by a scalar, and diagrammatic and column representations of vectors; **{use vectors to construct geometric arguments and proofs}**

Weeks 5 and 6: Algebraic Reasoning

Students develop their algebraic reasoning by looking at more complex situations. They use their knowledge of sequences and rules to made inferences, and Higher tier students move towards formal algebraic proof. Forming and solving complex equations, including simultaneous equations, is revisited. Higher tier students also look at solving inequalities in more than one variable.
National Curriculum content covered includes:

- make and test conjectures about the generalisations that underlie patterns and relationships; look for proofs or counter-examples; begin to use algebra to support and construct arguments **{and proofs}**
- deduce expressions to calculate the n^{th} term of linear **{and quadratic}** sequences
- solve two simultaneous equations in two variables (linear/linear **{or linear/quadratic}**) algebraically; find approximate solutions using a graph
- solve linear inequalities in one **{or two}** variable**{s}**, **{and quadratic inequalities in one variable}**; represent the solution set on a number line, **{using set notation and on a graph}**

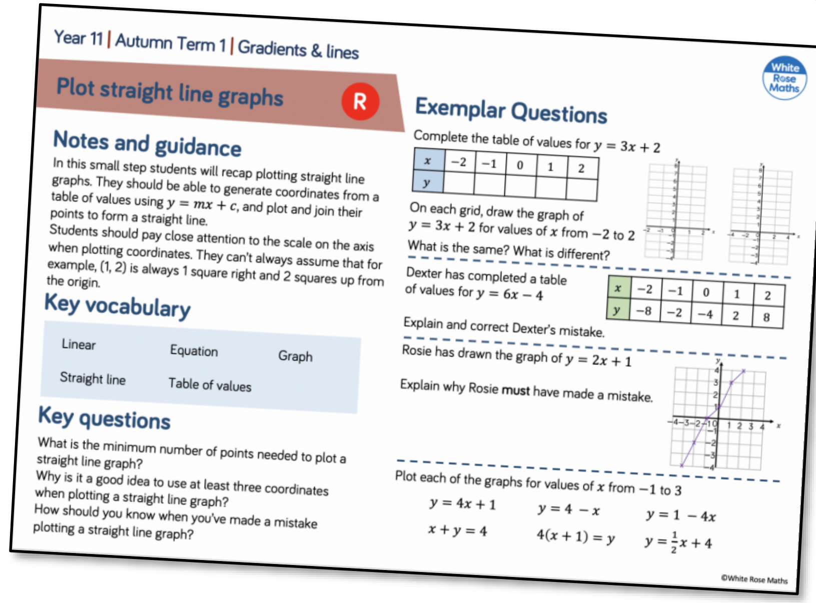
Why Small Steps?

We know that breaking the curriculum down into small manageable steps should help students to understand concepts better. Too often, we have noticed that teachers will try and cover too many concepts at once and this can lead to cognitive overload. We believe it is better to follow a "small steps" approach.

As a result, for each block of content in the scheme of learning we will provide a "small step" breakdown. ***It is not the intention that each small step should last a lesson – some will be a short step within a lesson, some will take longer than a lesson.*** We would encourage teachers to spend the appropriate amount of time on each step for their group, and to teach some of the steps alongside each other if necessary.

What We Provide

- Some ***brief guidance*** notes to help identify key teaching and learning points.
- A list of ***key vocabulary*** that we would expect teachers to draw to students' attention when teaching the small step.
- A series of ***key questions*** to incorporate in lessons to aid mathematical thinking.
- A set of questions to help ***exemplify*** the small step concept that needs to be focussed on.



Year 11 | Autumn Term 1 | Gradients & lines

Plot straight line graphs R

Notes and guidance

In this small step students will recap plotting straight line graphs. They should be able to generate coordinates from a table of values using $y = mx + c$, and plot and join their points to form a straight line.
Students should pay close attention to the scale on the axis when plotting coordinates. They can't always assume that for example, (1, 2) is always 1 square right and 2 squares up from the origin.

Key vocabulary

Linear, Equation, Graph, Straight line, Table of values

Key questions

What is the minimum number of points needed to plot a straight line graph?
Why is it a good idea to use at least three coordinates when plotting a straight line graph?
How should you know when you've made a mistake plotting a straight line graph?

Exemplar Questions

Complete the table of values for $y = 3x + 2$

x	-2	-1	0	1	2
y					

On each grid, draw the graph of $y = 3x + 2$ for values of x from -2 to 2
What is the same? What is different?

Dexter has completed a table of values for $y = 6x - 4$

x	-2	-1	0	1	2
y	-8	-2	-4	2	8

Explain and correct Dexter's mistake.

Rosie has drawn the graph of $y = 2x + 1$
Explain why Rosie **must** have made a mistake.

Plot each of the graphs for values of x from -1 to 3

$y = 4x + 1$ $y = 4 - x$ $y = 1 - 4x$
$x + y = 4$ $4(x + 1) = y$ $y = \frac{1}{2}x + 4$

©White Rose Maths

- These include reasoning and problem-solving questions that are fully integrated into the scheme of learning. Depending on the attainment of your students, you may wish to use some or all of these exemplars, which are in approximate order of difficulty. Particularly challenging questions are indicated with the symbol .
- For each block, we also provide ideas for key representations that will be useful for all students.

Some of the small steps are in **bold** and labelled with H to indicate this is Higher tier GCSE content. We would encourage teachers to use these with as many students as possible – if you feel your class can access any particular small step, then please include it in your planning. Steps that review content covered at Key Stage 3 are labelled R .

Algebraic Reasoning

Small Steps

- Simplify complex expressions
- Find the rule for the n^{th} term of a linear sequence (R)
- **Find the rule for the n^{th} term of a quadratic sequence** (R) (H)
- Use rules for sequences
- Solve linear simultaneous equations (R)
- **Solve simultaneous equations with one quadratic** (R) (H)
- **Formal algebraic proof** (H)
- **Inequalities in two variables** (H)

(H) denotes Higher Tier GCSE content

(R) denotes 'review step' – content should have been covered at KS3

Simplify complex expressions

Notes and guidance

Here students have an opportunity to revise algebraic notation and the rules for collecting like terms and indices. Answers could be checked by substitution as well if desired. Students should be aware of the associated language but may need a reminder of the word coefficient. Using the precise language makes it much easier to explain the simplifying process, particularly with division.

Key vocabulary

Term	Expression	Simplify
Power	Index	Coefficient

Key questions

How do you know when an expression is in its simplest form?

How does simplifying expressions relate to the laws of indices?

How many expressions can you find that simplify to (e.g.) $24x^2y^3$?

Exemplar Questions

Simplify the expressions, giving the answers as a single term.

$6 \times a$	$a \times 6$	$a \times 6a$	$6a \times b$
$6a \times 2b$	$6a \times 2ab$	$6a^2 \times 2ab$	$6a^2b \times ab^2$

Simplify the expressions, giving the answers as a single term.

$\frac{18xy}{3}$	$\frac{18xy}{3x}$	$\frac{18xy}{3y}$	$\frac{18xy}{3xy}$	$\frac{18xy}{8x}$	$\frac{18x^3y}{8x^2y}$	$\frac{18x^3y}{9x^2y^2}$

Explain the difference between these divisions.

- $12a^2 \div 6$
- $12a^2 \div 6a$
- $6a^2 \div 12a$
- $6a \div 12a^2$

Rosie thinks $(3x^2)^3$ can be rewritten as $9x^6$

Explain why Rosie is wrong.

Write these expressions without brackets.

- $(2a^3)^2$
- $(5p^3)^3$
- $(6a^2b^4)^2$
- $(2ab)^3 \times 3a^3b$

Which of the expressions cannot be simplified?

$xz + yz + xy$	$yx + xy$	$3yx + 3xz$
$3yx \times 3xz$	$yz + \frac{1}{2}zy$	$3yx \div 3xz$

n^{th} term of a linear sequence R

Notes and guidance

Students are very familiar with linear sequences and this review step is a reminder of previous learning. Encourage students to check their answers by substituting several values for n. To extend challenge, students could look at patterns and explain how the values of a and b in the rule $an + b$ relate to the pattern. Beware of the misconception that, for example, a sequence with a constant difference of 4 has the rule $n + 4$

Key vocabulary

Linear	Sequence	Non-linear
Term	Expression	Coefficient

Key questions

What's the connection between the coefficients of n in the rule for a linear sequence and the behaviour of the sequence? How would this change if n were negative?

When you know the coefficients of n in the rule for the n^{th} term of a sequence, explain how you find the rest of the rule.

Exemplar Questions

Match each sequence to the rules for the n^{th} term.

3, 6, 9, 12 …	$3n + 1$
1, 4, 7, 10 …	$3n$
4, 7, 10, 13 …	$3n - 2$

A sequence starts 10, 12, 14, 16 …

Dexter says the rule for the n^{th} term of the sequence is $n + 2$

Explain why Dexter is wrong, and find the correct rule.

Draw the next pattern in the sequence.

Find the rule for the number of counters in the n^{th} pattern.

- What is the 100th even number?
- What is the 100th odd number?
- Find rules for the n^{th} even number and the n^{th} odd number.

Here are the first five terms of an arithmetic sequence.

2 8 14 20 26

- Find an expression, in terms of n, for the n^{th} term of the sequence.
- Find an expression, in terms of n, for the $(n + 1)^{th}$ term of the sequence.

n^{th} term - quadratic sequence H

Notes and guidance

Higher tier students should have studied this in Year 10, so this step serves as revision. Students may try to apply the technique for finding the rule of a linear sequence, and should be encouraged to check their rule works by substitution of more than one value of n, and to identify the type of sequence before embarking on a solution.

Key vocabulary

Linear	Non-linear	Quadratic
Constant	Difference	Second difference

Key questions

What's the same and what's different about linear and quadratic sequences?

How do you find first and second differences? Do they relate to the coefficients of n in the rule for the n^{th} term of a sequence?

Exemplar Questions

Classify each sequence as linear, quadratic, or neither, explaining how you know.

1	3	4	7	11	…
18	15	12	9	6	…
8	13	20	29	40	…
6	5	2	-3	-10	…
1	2	4	8	16	…

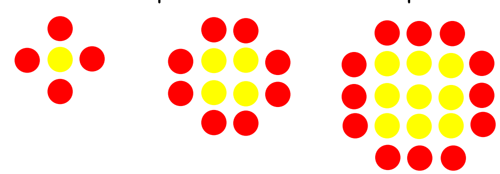
Explain how you can match the sequences to the rules without working out the rules or substitution.

4, 11, 22, 37, 56 …	$n^2 + 4n - 3$
2, 9, 18, 29, 42 …	$2n^2 + n + 1$
4, 7, 10, 13 …	$\frac{1}{2}(n^2 + n + 6)$

Find the rules for the n^{th} term of the sequences.

- 4, 12, 22, 34, 48 …
- 1, 5, 13, 25, 41 …

Draw the next pattern in the sequence.



Find the rule for the number of counters in the n^{th} pattern. Explain why the rule works.

Use rules for sequences

Notes and guidance

Here students build on their learning and use reasoning to determine, for example, whether a term is a member of a sequence or not. Students may need support to realise that the questions can be approached through forming and solving equations and inequalities rather than trying to list an excessive amount of terms. Here students can be reminded about geometric and Fibonacci sequences as well.

Key vocabulary

Linear	Geometric	Quadratic
Fibonacci	Equation	Inequality

Key questions

How could forming an equation help here? What is the variable? What does it mean if the solution to the equation is not an integer?

What's the difference between a linear sequence and a geometric sequence?

How can you identify a Fibonacci sequence?

Exemplar Questions

Show that 27 is a term in all these sequences.

15, 17, 19, 21 …	−1, 4, 3, 7, 10 …	$5n + 2$	$90 - 7n$
$8n - 5$	864, 432, 216, 108 …	$n^2 + 2$	

Here are the first five terms of an arithmetic sequence.

2 8 14 20 26

Find an expression, in terms of n, for the n^{th} term of the sequence.
Hence determine whether 150 is a term in the sequence.
Which term in the sequence is the first to exceed 300?

The n^{th} term of a sequence is given by $n^2 + 9$
Write down the first three terms of the sequence.
Which term of the sequence is 109?

Whitney says that all terms in the sequence $4n + 2$ are even.
Determine whether Whitney is correct. Explain why.

A Fibonacci sequence starts $a, b, a + b$.
Which of these terms are in the sequence?

$a + 2b$ $2a + b$ $3a + 5b$

Annie thinks the terms of the sequence given by the rule $4n + 6$ will be double the terms of the sequence given by the rule $2n + 3$
Do you agree with Anne?

Linear Simultaneous Equations R

Notes and guidance

Students explored solving a pair of linear simultaneous equations in Autumn Year 10, so this provides a timely reminder. Students could explore different approaches, including substitution where appropriate as opposed to relying only on the elimination method. Students may also need help to see when it is necessary to form simultaneous equations.

Key vocabulary

Coefficient	Linear	Simultaneous
Eliminate	Substitute	

Key questions

What does simultaneous mean?

How can you check your solutions to a simultaneous equations question? Does it matter which equation you use?

What clues in a question suggest you need to form a pair of simultaneous equations?

Exemplar Questions

Write down a pair of simultaneous equations with solutions.

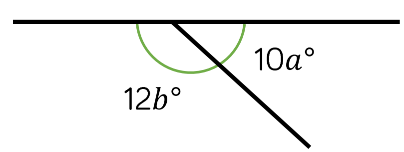
- $x = 5$ $y = 3$
- $x = 5$ $y = -3$
- $x = -5$ $y = -3$

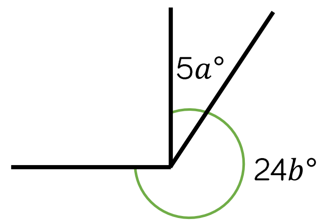
Compare your strategy with a partner's.

What's the same and what's different about these three pairs of simultaneous equations? How would you go about solving each pair?

$x + y = 60$ $x - y = 8$	$x + y = 60$ $2x + y = 73$	$x + y = 60$ $y = 2x$

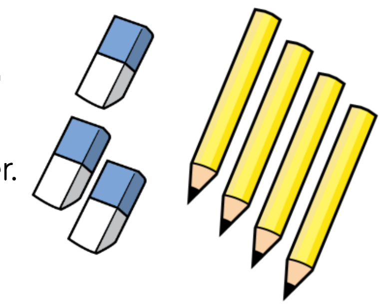
Use an algebraic method to work out the missing angles.





Two pencils and three erasers cost £2.85
Three pencils and two erasers cost £3.30

Eva wants to buy six pencils and an eraser.
She has £3
Does Eva have enough money?



Simultaneous Equations

H

Notes and guidance

Here Higher tier students revisit solving a pair of simultaneous equations where one is quadratic. Since first meeting this in Year 10, students have learnt other methods of solving quadratic equations and studied many other areas of maths. This means they can now practise using the quadratic formula as well as factorisation and work through equations in other contexts e.g. where a line meets a curve.

Key vocabulary

Coefficient	Linear	Simultaneous
Eliminate	Substitute	

Key questions

What's the same and what's different about solving two linear simultaneous equations and a pair where one is linear and the other quadratic?

How can you solve a quadratic equation when you cannot factorise?

Exemplar Questions

Use graphing software to draw the graphs of $x^2 + y^2 = 25$ and $y = x + 1$

- Find the coordinates of the points where the graphs meet.
- Verify algebraically that these points are the solutions to the simultaneous equations.

$$x^2 + y^2 = 25, \qquad y = x + 1$$

- Use the graphing software to draw other straight line graphs with gradient ± 1 that meet $x^2 + y^2 = 25$ at points with integer coordinates.
- Check your results using substitution and by solving pairs of simultaneous equations.

Compare solving these two pairs of simultaneous equations.

$$x^2 - 4x - 3y = 0 \qquad x + y = 4$$

$$x^2 - 4x - 3y = 0 \qquad x + y = 5$$

What's the same and what's different?

- Explain how you can tell, without drawing the graphs or doing any calculations, that the pair of simultaneous equations

$$y = 3x + 5 \qquad y = x^2 + 4x + 5$$

have a solution when $x = 0$?

- Find the other solution.

Will a pair of simultaneous equations where one is linear and the other quadratic always have 2 solutions? Justify your answer.

Formal algebraic proof

H

Notes and guidance

Students need to be comfortable with algebraic manipulation to complete formal algebraic proof, so it is worth starting with a reminder of this. Students can use cubes or bar models to appreciate than $2n$ is even for integer n and that $2n + 1$ and $2n - 1$ are both odd. Likewise they should know e.g. $5k$ is a multiple of 5 They should also appreciate the meaning of the word counterexample and how to show a conjecture is false.

Key vocabulary

Proof	Demonstration	Counterexample
Justify	Even	Odd

Key questions

Why isn't a list of examples a proof?

If k is an integer, tell me some numbers (e.g.) $12k$ must be a multiple of.

If n is even, what can you say about $n + 1$? $n + 2$? $2n + 3$?

Exemplar Questions

By expanding the brackets, explain why $3(9x + 6) \equiv 9(3x + 2)$

Find some other expressions that are equivalent to $6(8x + 4)$

---

n is a positive integer. Which of these expressions must represent an even number, which must be odd and which could be either?

$2n$	$3n$	$4n$	n^2
$2n + 1$	$2n - 1$	$3n + 1$	$4n + 6$

Would any of your answers change if you knew n were

- even
- odd?

Investigate other expressions of the form $an + b$. Can you generalise?

Here is Dexter's proof that the sum of any two even numbers is even.

$2m = 2 \times m$ is even
$2m + 2m = 4m$
$4m = 2 \times 2m$, so it's even too.

Explain why Dexter's proof does not cover all pairs of even numbers and write a correct proof.
Show also that the sum of two odd numbers is also even.

- Explain why $8k$ must be a multiple of 8
- What is the next odd number after $2n + 1$?
- Prove that the difference between the squares of two consecutive odd numbers is always a multiple of 8

Inequalities in two variables

H

Notes and guidance

Here students explore inequalities in more than one variable, using a graphical approach. You may wish to start by exploring which side of a single line such as $y = x$ is the region $y > x$ and which is the region $y < x$. You may also need to remind students about the equations of lines parallel to the axis. The "cover-up" method is a quick way of finding where equations like $2x + 3y = 12$ meet the axes.

Key vocabulary

Inequality	Region	Satisfy
Equation	Integer	Test

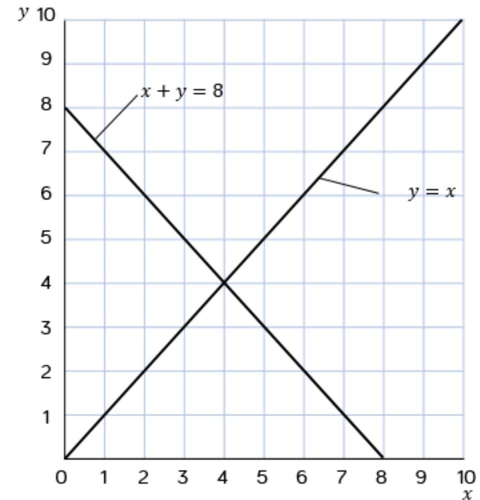
Key questions

How can you use the coordinates of a point in a region to test what inequalities it satisfies?

How do you choose a region when there is more than one condition to be satisfied?

What's the difference between finding a region and finding integer values that satisfy a set of inequalities?

Exemplar Questions



On separate copies of the grid and graphs shown, shade the regions where

- $y > x$
- $x + y < 8$
- $y > x$ and $x + y < 8$
- $y < x$ and $x + y < 8$
- $y < x$ and $x + y > 8$

Use another copy of the grid, draw suitable lines to find the integer values of x and y that satisfy all three of the inequalities $y > 4, x < 3$ and $x + y < 8$

Where does the line $5x + 4y = 20$ meet the axes?
What's the same and what's different about the line $5x - 4y = 20$?

Draw a pair of axes from −6 to 6 in both directions.
By drawing suitable graphs, mark with a cross the integer values of x and y that satisfy all the inequalities.

$3y + 2x < 12 \qquad y < 2x + 1 \qquad 0 < y < 3$